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SOLUTION OF MINIMUM PROBLEMS OF THE AIRFOIL THEORY

By Karl Nickel

Translated by B. R. Briggs, Ames Aero. Lab.

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Translated by B. R. Briggs
Ames Aeronautical Laboratory
Moffett Field, Calif.

The third fundamental problem of Prandtl's lifting line is extended to the case of a finite number of accessory conditions and the solution is given. Three examples are given demonstrating the practical occurrence of such problems.

1. PRESENTATION OF THE PROBLEM

In der Tragflügeltheory I of "Vier Abhandlungen"[1] L. Prandtl gives as the third fundamental problem the following minimum problem (on p. 28) "The total lift and wingspan are given and so are ρ and v; the distribution of lift over the span such that the drag is minimum is sought."

With help from the formula stated in the previously cited book (p. 27) the mathematical statement of the problem in which a coordinate

[1] L. Prandtl and A. Betz: Vier Abhandlungen zur Hydrodynamik und Aerodynamik. Neudruck aus den Verhandlungen des III. Internationalen Mathematiker - Kongresses zu Heidelberg und aus den Nachrichten des Gesellschaft der Wessenschafton zu Göttingen 1927.

system such that the wing extends from +1 to −1 is used is[2]

$$w(x) = \frac{1}{4\pi} \fint_{-1}^{1} \frac{d\Gamma(y)}{dy} \frac{dy}{x-y} \tag{1}$$

where $\Gamma(-1) = \Gamma(+1) = 0$ (here $\Gamma(x)$ is the local lift density, $w(x)$ is the induced downwash at the point x on the wing). The function $\Gamma(x)$ may be defined in the interval (+1, −1) such that

$$\int_{-1}^{1} \Gamma(x)\, w(x)\, dx \tag{2}$$

becomes a minimum under the associated condition

$$\int_{-1}^{1} \Gamma(x)\, dx = A \tag{3}$$

(A = arbitrary total lift).

This statement of the problem hasbeen extended by M. Munk[3] to the case of the lift arbitrarily directed and distributed in space, and solved generally.

One can also generalize the above minimum problem to another kind in which instead of the condition (3) another – and perhaps several – side conditions are prescribed. Three examples will make this clear.

(a) A flying airfoil shall, by use of a small aileron deflection, be in a flat curve. What form must the lift distribution due to the incremental aileron deflection combined with the symmetrical distribution

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[2]By the symbol $\fint$ shall be understood the Cauchy principle part of the integral.

[3]M. Munk: Isoperimetriche Aufgaben aus der Theory des Fluges. Inaugural-Dissertation. Göttingen 1919.

have in order that the increase in induced drag shall remain as small as possible? If one knows this distribution then one can approximate it with a suitable control-surface shape to eliminate so much aileron deflection loss. The question therefore becomes: sought is that distribution of lift which makes the induced drag (2) a minimum if the rolling moment

$$\int_{-1}^{1} x\,\Gamma(x)\,dx \tag{4}$$

has a prescribed value. The use of equation (1) is exactly correct only in the first instant of the aileron deflection, as long as only small turning and rolling motions exist, since only **then** are the trailing vortex lines straight. Nevertheless, one can still apply the solution of the formulated problem as an approximate solution for small turning and rolling velocities.

Another example which leads to the same problem, and in which these difficulties do not appear was communicated to me in a friendly manner by Prof. Dr. Prandtl. It is this, the question of the lift distribution of least drag for a wing with eccentrically applied load generated by a rolling moment.

(b) The solution of the third fundamental problem of Prandtl is written (see (1) page 32) in the case of the above formulation:

$$\Gamma(x) = \frac{2A}{\pi} \sqrt{1-x^2}$$

that is, the lift is distributed in a kind of half-ellipse over the span (for example, the solid line in figure 1). If one now changes $\Gamma(x)$ in the manner of the broken line where A in (3) retains its value, then it will change the induced drag only slightly (in the neighborhood of its minimum). The bending moment about the wing root

$$\int_0^1 x\,\Gamma(x)\,dx$$

which for symmetrical lift distributions can be replaced by

$$\int_{-1}^{1} |x|\,\Gamma(x)\,dx \qquad (5)$$

would, however, become smaller (if a factor 1/2 is neglected).

For a wing in free flight the following is significant: owing to the diminishing loading the wing can be built lighter. Since, however, in stationary horizontal flight lift and weight are equal, this means that the total lift in (3) becomes smaller.

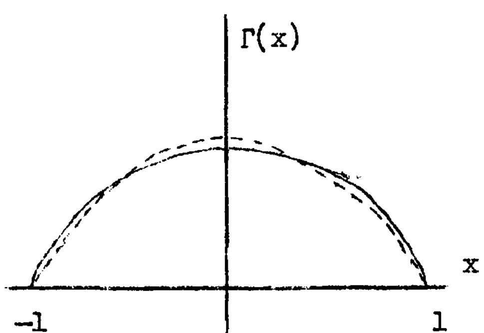

Fig. 1

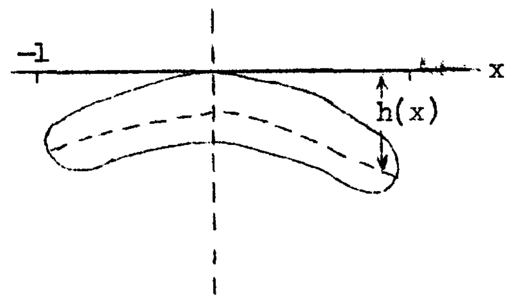

Fig. 2

If one replaces $\Gamma(x)$ by $\lambda \cdot \Gamma(x)$ then one varies lift (3) and bending moment (5) proportional to λ, while the induced drag is proportional to λ^2. One will also, with a lift distribution which at the wingtips has somewhat smaller values than the elliptic distribution with equal lift, expect somewhat better drag characteristics. It is true it must be noted, that the dependence of the total loading on the bending moment in the case of the usual wing construction is very slight, so that the effect described above is very small.

This consideration leads to the following problem statement: it shall be to make the drag (2) a minimum under the two conditions that lift (3) and bending moment (5) have prescribed values.

(c) If one considers a wing that is curved in the direction of flight (see figure 2), then in the case of the third fundamental problem of wing theory[4] the pitching moment as well as the total lift would be held constant. One can now require that the pitching moment shall result exclusively from the lifting force on the wing. (Pitch stabilization by means of warped wings as in the flight of birds.) Let $h(x)$ be the distance from the x-axis to the line of centers of pressure (the broken line in figure 2). Then state the requirement that

$$\int_{-1}^{1} h(x)\, \Gamma(x)\, dx = M \tag{6}$$

(pitching moment prescribed equal to M) shall hold and the above problem reads: the lift distribution $\Gamma(x)$ is sought such that (2) shall be a minimum under the conditions (3) and (6). It may be stated that in the general case an arbitrarily shaped wing with line of centers of pressure not coincident with the forward aerodynamic centers in the two-dimensional problem (T/4 line) will be given. The problem of

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[4]Indeed (see (1) page 25) the illustration of the lifting line cannot be applied in this case, as M. Munk((3) page 21) has nevertheless shown it is sufficient in the case of drag considerations to calculate only the two-dimensional problem of a wing not curved in the direction of flight. $\Gamma(x)$ is consequently the projection of the lift on a plane perpendicular to the line of flight.

determining this line for an arbitrarily prescribed wing area involves great difficulty and as far as I know it has not up to the present been solved. One can, nevertheless, approximate the line of centers of pressure by the $T/4$ line as long as the wing does not vary too much from a flat wing.

It is natural now, on the basis of this example, to extend the above formulated problem to finitely many prescribed subsidiary conditions which are linear in $\Gamma(x)$. One seeks to determine $\Gamma(x)$ in the interval (-1, 1) such that

$$\int_{-1}^{1} \Gamma(x)\, w(x)\, dx \tag{2}$$

will be a minimum under the N side conditions

$$\int_{-1}^{1} \Gamma(x)\, h_n(x)\, dx = A_n \quad (n=1, \ . \ . \ . \ , N) \tag{7}$$

$\Gamma(-1) = \Gamma(+1) = 0$ $h_n(x)$ and A_n arbitrarily prescribed, naturally for $A_m = 0$ only $h_m(x) \equiv 0$ is allowable for $N = 1$, $h_1(x) \equiv 1$, $A_1 = A$ the L. Prandtl formulated problem follows as a particular case just as can the examples a, b, and c be obtained by analogous specialization.

2. TRANSFORMATION OF THE PROBLEM

By mathematical manipulation of the problem the following transformations will be undertaken:

Let

$$x = \cos s \ , \ y = \cos t$$

$$4w(x) \ \sqrt{1-x^2} = f(s), \ \Gamma(x) = z(s)$$

and use the notation $h_n(x)\sqrt{1-x^2} = h_n(s)$, $(n=1, \ldots, N)$. Using these there results[5] from (1), (2) (neglecting a factor 1/4)

$$f(s) = \frac{1}{\pi} \oint_0^\pi \frac{dz(t)}{dt} \frac{\sin s}{\cos t - \cos s} \, dt \tag{1a}$$

$$\int_0^\pi z(s) f(s) \, ds \tag{2a}$$

$$\int_0^\pi z(s) h_n(s) \, ds = A_n \quad (n=1, \ldots, N) \tag{7a}$$

In the case of known $f(s)$ (1a) is a Fredholm integral equation of the first kind. Its solution is known to be[6]

$$\frac{dz(s)}{ds} = \frac{1}{\pi} \oint_0^\pi f(t) \frac{\sin t}{\cos s - \cos t} \, dt$$

where by integration there results

$$z(s) = \frac{1}{\pi} \int_0^\pi f(t) \log \frac{\sin \frac{s+t}{2}}{\sin \left| \frac{s-t}{2} \right|} \, dt \tag{8}$$

[5] The assumptions under which the integrals and infinite series exist and under which the applied transformations (interchange of order of integration, interchange of integration and summation) are allowed, will be considered in a detailed work which will appear shortly under the title "Solutions of Some Special Minimum Problems" in the Math. Zeit. Moreover a second class of subsidiary conditions will be taken into consideration. Addition by the referee: It has appeared in the meantime in the Math. Zeit., Bd. 53 (1950) pages 21–52.

[6] See perhaps K. Schröder: Über eine Integralgleichung erstn Art der Tragflügeltheory. Sitzungs-berichte der Preussischen Akademie der Wissenschaften XXX (1938)

For simplification write

$$\frac{1}{\pi} \log \frac{\sin \frac{s+t}{2}}{\sin \left|\frac{s-t}{2}\right|} = S(s,t)$$

and place (8) in (2a) and (7a), giving

$$\int_0^\pi \int_0^\pi f(s)f(t)\; S(s,t)\; ds\; dt \tag{2b}$$

$$\int_0^\pi \int_0^\pi f(s)h_n(t)\; S(s,t)\; ds\; dt = A_n \qquad (n=1, \ldots, N) \tag{7b}$$

Since from (1a) and (8) the functions $z(s)$ and $f(s)$ are clearly dependent on each other one can also regard $f(s)$ as the unknown. With this the problem to be solved is: determine $f(s)$ in the interval $(0,\pi)$ such that (2a) will be a minimum under the associated condition (7b).

For further simplification the following notation will be introduced:

$$(f,g) = \int_0^\pi \int_0^\pi f(s)g(t)\; S(s,t)\; ds\; dt \tag{9}$$

With the representation[7]

$$S(s,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin ns \sin nt$$

valid for $0 \leq s,\; t \leq \pi,\; s \neq t,$ it follows from (9) that

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[7]See, for example, G. Hammel, Integralgleichungen page 20. Berlin 1938 by S. Springer; also K. Jaeckel, Ermittlung einer Reihendarstellung des Kernes ln r in elliptichen Koordinaten. ZAMM, Bd. 30 (1950) page 186. Formula (16)

$$(f,g) = \int_0^{\pi}\int_0^{\pi} f(s)g(t)\ S(s,t)\ ds\ dt$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left\{\int_0^{\pi} f(s)\ \sin nsds\right\}\left\{\int_0^{\pi} g(t)\ \sin ntdt\right\}$$

or

$$(f,g) = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n}\, a_n\, b_n \tag{10}$$

with the Fourier sine coefficients

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(s)\ \sin nsds, \quad b_n = \frac{2}{\pi}\int_0^{\pi} g(t)\ \sin nt\ dt$$

$$(n=1,\ 2,\ .\ .\ .\ )$$

From (9) and (10), and since $S(s,t) = S(t,s)$ (f,g) possesses the following properties

$$\left.\begin{array}{ll} \text{(a)}\ (af,g) = a(f,g) & (a = \text{real number}) \\ \text{(b)}\ (f,g) = (g,f) & \\ \text{(c)}\ (f+g,h) = (f,h) + (g,h) & \\ \text{(d)}\ (f,f) > 0 \text{ for } f(s) \neq 0 & \end{array}\right\} \tag{11}$$

One also has the problem of finding a function $f(s)$ in the interval $(0,\pi)$ such that

$$(f,f) = \min.$$

holds under the N subsidiary conditions

$$(f,h_n) = A_n \quad (n=1,\ .\ .\ .\ ,\ N) \tag{12}$$

Here (f,g) is defined by (9) and has the properties (11)

3. SOLUTION OF THE PROBLEM

If the functions $h_n(s)$ are linearly dependent, then either there are superfluous conditions in (12), or they are contradicted among themselves, and they are generally not fulfilled.

Assumption: The functions $h_n(s)$ are linearly independent, the real numbers A_n (n=1, . . . , N) are arbitrarily chosen.

Contention: Then there is exactly one solution $f'(s)$ of the above minimum problem and this solution has the form

$$f'(s) = \sum_{n=1}^{N} A_n h_n(s)$$

with uniquely determined numbers A_n.

Proof: With the orthonormalization process of E. Schmidt one can find functions

$$H_n(s) = \sum_{n=1}^{n} C_{mn} h_m(s) \qquad C_{mn} \neq 0; \quad n=1, \ldots, N)$$

such that

$$(H_m \cdot H_n = C_{mn} = \begin{cases} 1 \text{ for } m=n \\ 0 \text{ for } m \neq n \end{cases}$$

When these equations are inverted the following forms result

$$h_n(s) = \sum d_m n \, H_m(s) \qquad (d_n n \neq 0; \; n=1, \ldots, N)$$

Now let

$$f'(s) = \sum_{n=1}^{N} C_n H_n(s)$$

With real coefficients C_n undetermined for the present, and obtain with (11)

$$(f', h_r) = \sum_{n=1}^{V} C_n (H_n, h_r) = \sum_{n=1}^{N} C_n \sum_{m=1}^{r} d_{mr} (H_n, H_n)$$

$$= \sum_{n=1}^{N} C_n \sum_{m=1}^{r} d_{mr}\, e_{nm} = \sum_{n=1}^{r} C_n d_{nr}$$

for $r=1, \ldots, N$

Now, if $\sum_{n=1}^{r} C_n d_{nr} = A_r$ $(r=1, \ldots, N)$ shall hold, then the C_n are clearly determined. With the numbers thus chosen f' fulfills the conditions (12), furthermore $f'(s)$ obtains the general form

$$f'(s) = \sum_{m=1}^{N} C_m H_m(s) = \sum_{m=1}^{N} C_m \sum C_{nm} h_n(s)$$

$$= \sum_{n=1}^{N} h_n(s) \sum_{m=n}^{N} C_m C_{nm} = \sum_{n=1}^{N} a_n h_n(s)$$

It remains to show that f' also makes (f,f) a minimum. Now an arbitrary function $f(s)$ can always be written in the form

$$f(s) = f'(s) + k(s) \qquad (\text{namely with } k(s) = f(s) - f'(s))$$

Then if

$$(f,h_n) = (f'+k,h_n) = (f',h_n) + (k,h_n)$$
$$= A_n + (k,h_n) \qquad (n=1, \ldots, N)$$

$f(s)$ satisfies conditions (12) exactly, if

$$(k,h_n) = 0 \text{ for } n=1, \ldots, N$$

For the totality of functions $f(s)$ which fulfill the conditions (12) the following holds, according to (11)

$$(f,f) = (f'+k,f'+k) = (f',f') + 2(f',k) + (k,k)$$

$$= (f',f') + 2\sum_{n=1}^{N} a_n(h_n,k) + (k,k)$$

$$= (f',f') + (k,k) \geq (f',f')$$

where the equality sign holds only for $k(s) \equiv 0$. The uniqueness of the coefficients a_n follow with (11d) from the linear independence of the functions $h_n(s)$. Therewith the contention is proved.

If one returns to the original notation one finds the following conclusions.

4. CONCLUSION

The functions $h_n(x)$ and the real numbers A_n $(n=1, \ldots, N)$ have given arbitrary values. The function $\Gamma(x)$ shall be so defined in the interval $-1 \leq x \leq +1$ that $\Gamma(-1) = \Gamma(+1) = 0$ and that under the substitution

$$w(x) = \frac{1}{4\pi}\int_{-1}^{1} \frac{d\Gamma(y)}{dy}\,\frac{dy}{x-y} \qquad (1)$$

the expression

$$\int_{-1}^{1} \Gamma(x)w(x)\, dx \tag{2}$$

is minimized under the subsidiary conditions

$$\int_{-1}^{1} \Gamma(x)h_n(x)dx = A_n \qquad (n=1, 2, \ldots, N) \tag{7}$$

If a function $\Gamma(x)$ is given generally, such that the N subsidiary conditions are satisfied, then this minimum problem has exactly one solution, which assumes the form

$$\Gamma(x) = \sum_{n=1}^{N} a_n \int_{-1}^{1} h_n(y) \log \left| \frac{1-xy+\sqrt{(1-x^2)(1-y^2)}}{x-y} \right| dy$$

with real coefficients a_n. If the functions $hn(x)$ are linearly independent then the minimum problem always has a solution and the coefficients are uniquely determined.

5. APPLICATION

In the following table some simple functions $h(x)$ are related to the appropriate function

$$I = \int_{-1}^{1} h(y) \log \left| \frac{1-xy+\sqrt{(1-x^2)(1-y^2)}}{x-y} \right| dy$$

h(x)	
1	$\pi \sqrt{1-x^2}$
x	$\frac{\pi}{2} x \sqrt{1-x^2}$
x^2	$\frac{\pi}{6} (2x^2+1) \sqrt{1-x^2}$
x^3	$\frac{\pi}{8} x(2x^2+1) \sqrt{1-x^2}$
$\begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$	$2x \log \frac{1 + \sqrt{1-x^2}}{\|x\|}$
$\|x\|$	$x^2 \log \frac{1 + \sqrt{1-x^2}}{\|x\|} + \sqrt{1-x^2}$
$x \|x\|$	$\frac{2}{3}\left(x^3 \log \frac{1 + \sqrt{1-x^2}}{\|x\|} + x \sqrt{1-x^2}\right)$
$x^2 \|x\|$	$\frac{1}{12}\left(6x^4 \log \frac{1 + \sqrt{1-x^2}}{\|x\|} + (6x^2+1) \sqrt{1-x^2}\right)$
$\begin{cases} 1 & xo < x \leq 1 \\ 0 & -1 \leq x < xo \end{cases}$	$(xo-x) \log \left\| \frac{1 - xox + \sqrt{(1-xo^2)(1-x^2)}}{xo-x} \right\| + \sqrt{1-x^2} \cos^{-1} xo$

Thereby one finds for the first pair of examples stated at the outset the solutions:

Subsidiary Condition	Solution
(a) $\int_{-1}^{1} \Gamma(x)\, x dx = R$	$\Gamma(x) = \frac{8R}{\pi}\, x\, \sqrt{1-x^2}$

(b)
$$\left\{ \begin{array}{l} \int_{-1}^{1} \Gamma(x)\, dx = A \\ \int_{-1}^{1} \Gamma(x)\, |x|\, dx = H \end{array} \right\}$$

$$\Gamma(x) = 3\left(\frac{2A}{\pi} - H\right)\sqrt{1-x^2} + 3\left(3H - \frac{4A}{\pi}\right) x^2 \log \frac{1+\sqrt{1-x^2}}{|x|}$$

In figure 3 this solution of example A is drawn. Figure 4 shows several curves from the many solutions of example (b) for A = Const., H variable (for $3\pi H = 4A$ the half elipse is seen to be a special case of (b)).

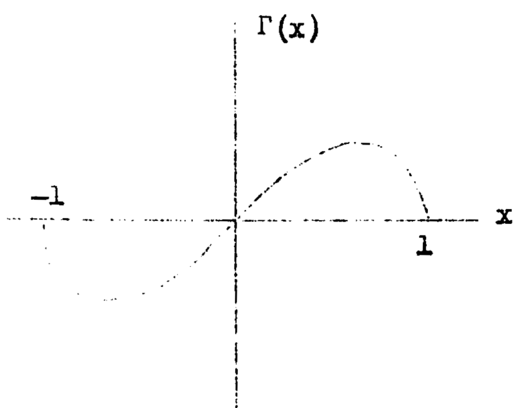

Fig. 3 to example (a)

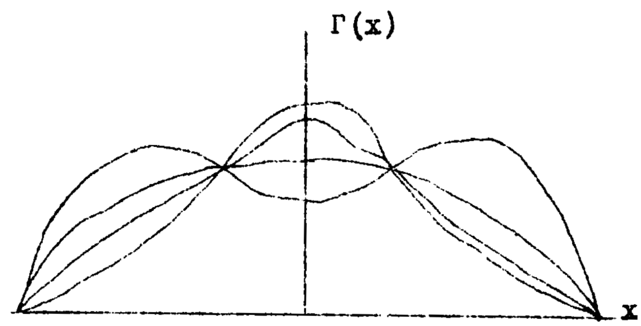

Fig. 4 to example (b)